Comment on "Two-finger selection theory in the Saffman-Taylor problem"

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(Received 7 February 2000; published 21 March 2001)

It is pointed out that the two-parameter family of solutions for the Saffman-Taylor problem recently studied by Magdaleno and Casademunt [Phys. Rev. E **60**, R5013 (1999)] does not correspond to two fingers moving in a Hele-Shaw cell with the channel geometry, as was implied in their paper. It is thus clarified that their solution, while correctly describing a periodic array of axisymmetric fingers in an unbounded Hele-Shaw cell, gives rather a central finger flanked by two half-fingers when restricted to a channel with impermeable rectilinear walls. The correct four-parameter family of exact solutions for two unequal fingers in such a channel is presented for the zero surface tension case.

DOI: 10.1103/PhysRevE.63.043101 PACS number(s): 47.54.+r, 47.20.Ma, 47.20.Hw, 47.20.Ky

Recently, Magdaleno and Casademunt [1] reported a selection theory for the Saffman-Taylor problem with multiple fingers. The starting point of their analysis was a two-parameter family of exact solutions for the zero surface tension case, which they claimed described two unequal fingers with different relative widths λ_1 and λ_2 moving steadily in a Hele-Shaw channel. They then carried out a standard solvability analysis [2] and found that, for vanishingly small surface tension, an infinite discrete set of values for the parameters $\lambda=\lambda_1+\lambda_2$ and $p=\lambda_1/\lambda$ is selected out of the twice continuous-degenerate solution. The aim of the present Comment is twofold: (i) to clarify that the solutions studied by Magdaleno and Casademunt [1] do not actually correspond to two fingers in a channel with rigid walls, and (ii) to present the correct four-parameter family of exact solutions for two unequal fingers in a rectilinear Hele-Shaw channel when surface tension is neglected.

I begin by considering the exact solution studied by Magdaleno and Casademunt [1]. Their solution was obtained via a conformal mapping $z=f(w,t)$ that maps the interior of the unit circle in the w complex plane onto the viscous fluid region in the z plane, with the unit circle $w=e^{i\phi}$ being mapped onto the interfaces. As noted in Ref. [1], the mapping function $f(w,t)$ contains a logarithmic singularity at the origin, and to ensure its analyticity within the flow domain in the w plane a branch cut must therefore be inserted in the interior of the unit circle. The location of this branch cut, however, was not explicitly indicated by Magdaleno and Casademunt [1], and this omission might lead to confusion regarding the actual geometry described by their solution. It should thus be pointed out that owing to the symmetry adopted (the fingers are axisymmetric) the branch cut must be inserted along the imaginary axis: $w=-i\rho$, $0\leqslant\rho\leqslant 1$. The two sides of this slit are respectively mapped onto two horizontal lines, say $y=\pm\pi$, which thus define a rectangular period cell of width 2π in the z plane. Within this unit cell the interface is described by the following equation [1]:

$$x_0(\phi)=(1-\lambda)\ln(2|\sin\phi-\cos p\pi|), \tag{1}$$

supplemented with the condition $y_0(\phi)=-\lambda\phi+c(\phi)$, where $c(\phi)$ is a piecewise constant function. (Here the definition of $c(\phi)$ differs by a factor of λ from that used by Magdaleno and Casademunt [1].) In view of the existence of the aforementioned slit in the w plane, the parameter ϕ takes value in three intervals, as follows. The first interval $(-\pi/2,\phi_1)$, where $\phi_1=(\pi/2)(1-2p)$, corresponds to a half finger touching the upper cell wall; the second interval (ϕ_1,ϕ_2), with $\phi_2=(\pi/2)(1+2p)$, gives a centerline symmetric finger; and the third interval $(\phi_2,3\pi/2)$ represents a half finger in contact with the lower wall; see Fig. 1. As ϕ moves from one interval to the next, the function $c(\phi)$ above jumps by the finite amount $\pi(1-\lambda)$, which accounts for the width of the fluid region between adjacent fingers. Analytic continuation of the solution to the entire z plane is affected by successive reflections about the cell walls, thus generating a periodic array of fingers. Notice, however, that this extended solution does not include the case of two fingers in a channel with rectilinear impermeable walls. I also recall here that in Ref. [1] the angle ϕ was taken to vary within two intervals, namely, (ϕ_1,ϕ_2) and $(\phi_2,\phi_3=2\pi+\phi_1)$. Although this choice correctly describes the two finger shapes for the periodic solution, it should be noted that, rigorously speaking, the parameter ϕ in Eq. (1) runs over three intervals, as shown above.

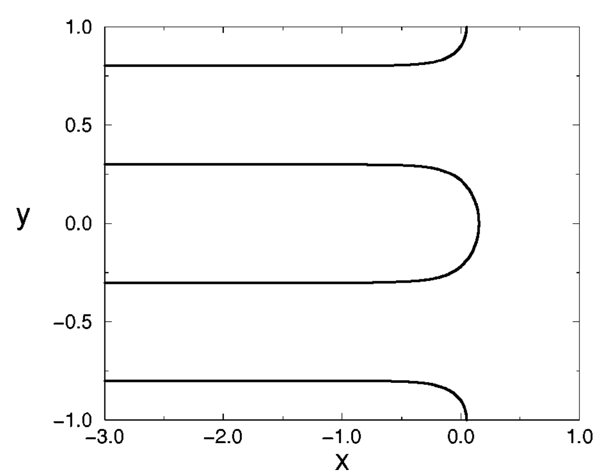

FIG. 1. Multifinger configuration corresponding to Eq. (1) with $\lambda=1/2$ and $p=0.6$. Here the solution has been rescaled by a factor of π so as to place the channel rigid walls at $y=\pm 1$.

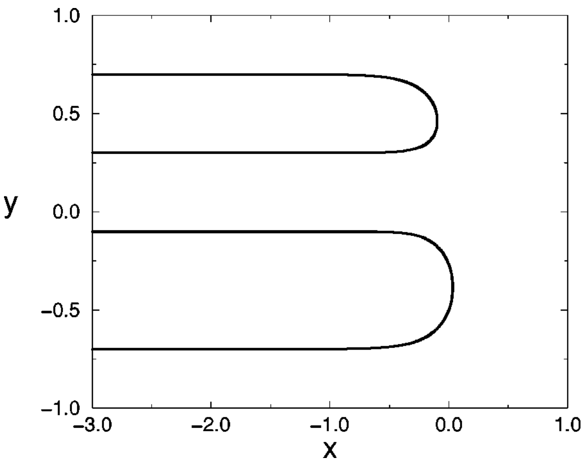

FIG. 2. Two-finger solution with $\lambda = 1/2$, $p = 0.6$, $q = r = 0.3$. The same rescaling as in Fig. 1 was applied.

Now I present the correct solution for two unequal fingers in a rectilinear Hele-Shaw channel when surface tension is neglected. This two-finger solution can be easily obtained from the family of exact solutions previously reported by the author [3] that describe an arbitrary number N of fingers advancing with the same velocity in a Hele-Shaw channel in the absence of surface tension. Thus, to obtain the two-finger solution one must simply set $N = 2$ in the general formula given in Eq. (8) of Ref. [3]. Doing this and performing some simplification, one then finds that the finger interfaces are described by

$$x(\theta) = 2(1-\lambda)[q \ln(1-\cos\theta) + r \ln(1+\cos\theta) + (1-q-r)\ln|\cos\theta + \cos p\pi|], \tag{2}$$

together with $y(\theta) = -2\lambda\theta + c(\theta)$. Here λ and p are as defined before, and the additional parameters q and r are the asymptotic widths of the respective fluid regions between the upper and lower fingers and the channel walls relative to the total width occupied by the fluid. With this notation, the relative width of the fluid region separating the two fingers is thus given by $1-q-r$. The parameters q and r range from 0 to 1 and must obviously satisfy the condition $q + r \leqslant 1$. The two fingers correspond to the intervals $(0, \theta_1)$ and (θ_1, π), respectively, where $\theta_1 = \pi(1-p)$. On these two intervals the function $c(\theta)$ takes the constant values $\pi[1-2q(1-\lambda)]$ and $\pi[1-2(1-r)(1-\lambda)]$, respectively. In Fig. 2 it is shown a solution with $\lambda = 1/2$, $p = 0.6$, $q = r = 0.3$. [I note in passing that the particular solution given in Eq. (1) is simply a special case of the general solution (2) with $q = r = 0$. In this limit, for example, the solution shown in Fig. 2 would reproduce precisely the upper half of the channel seen in Fig. 1.]

As a concluding remark, I wish to emphasize that the solvability analysis carried out by Magdaleno and Casademunt [1] remains valid, albeit more of mathematical interest since the geometry described by their solutions (seen either as a periodic array of fingers in an unbounded cell or as a finger with two half fingers in a channel) is less relevant from an experimental viewpoint. It thus remains an open and interesting problem to extend the solvability analysis to include the general solution for two unequal fingers in a channel described in this Comment. On the basis of the results reported in Ref. [1], it is reasonable to expect that an infinite discrete set of the parameters λ, p, q, and r will be selected for vanishingly small surface tension. If this is indeed the case, it would be of particular interest to find out whether in the limit of zero surface tension the selected solutions will all converge to the equal-finger solution (as was the case in Ref. [1]) or whether unequal fingers may also survive.

This work was supported in part by Financiadora de Estudos e Pesquisas, Conselho Nacional de Desenvolvimento Científico e Tecnológico, and PRONEX under Grant No. 76.97.1004.00.

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